Multiparameter Binomial Sums

AMS Meeting - University of South Florida

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Outline

Binomial Coefficients - $\binom{n}{k}$

Two conventions:

1. **Combinatorial**: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of distinct ways to form a k-element subset from a set of size n.

2. **Complex Analysis and Special Functions**:

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)},$$

where Γ is the meromorphic continuation of the factorial (!) function.

Definition and Some Properties

Let

$$S_n(\alpha,\beta,\gamma,\lambda,\mu) = \sum_{k=0}^{n} \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}} \left[\frac{\binom{\gamma+n-1}{n-k}}{\binom{\gamma+\lambda+n-1}{n-k}}\right]^{\mu}$$

such that $\alpha+\beta,\ \gamma+\lambda \neq 0,-1,-2,\ldots$

Under some restrictions:

1. $\lim_{n\to\infty} S_n = \binom{\alpha+\beta-1}{\beta} / \binom{\lambda\mu+\alpha+\beta-1}{\beta}$,
2. S_n is completely monotone,
3. $S_n(\alpha,\beta+\lambda,\alpha,-\gamma,\mu)^{\nu} \cdot S_n(\beta,\alpha+\lambda,\beta,\gamma,\nu)^{\mu} \geq 1 \ \left[\frac{1}{\mu}+\frac{1}{\nu}=1\right]$.
4. S_n may be expressed as a hypergeometric function

Binomial Distribution - $\mathrm{B}(n, p)$

The **binomial distribution** is a discrete distribution with probability mass function

$$f(t \mid n, p) = \binom{n}{t} p^t (1-p)^{n-t}$$

for $t = 0, 1, \ldots, n$.

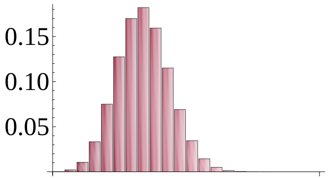

Figure 1: PMF of $\mathrm{B}\left(21, \frac{1}{3}\right)$.

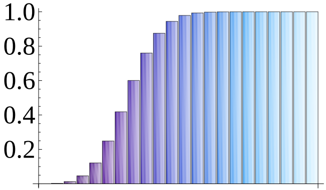

Figure 2: CDF of $\mathrm{B}\left(21, \frac{1}{3}\right)$.

Beta Function - $B(x, y)$

For $\Re(x), \Re(y) > 0$ the **beta function** is defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

which is directly related to the gamma function in that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

meaning for $x + y \neq 1$,

$$B(x, y) = \frac{1}{(x+y-1)\binom{x+y-2}{x-1}} .$$

Beta Distribution - Beta(α, β)

The **beta distribution** is a continuous distribution with pdf

$$f(t \,|\, \alpha, \beta) = \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha, \beta)}$$

for $t \in [0, 1]$.

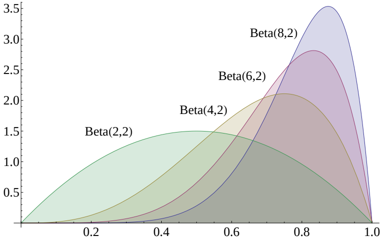

Figure 3: PDFs of various beta distributions.

Beta-Binomial Distribution - BB(α, β, n)

The **beta-binomial** distribution is a discrete distribution with pdf

$$f(t \,|\, \alpha, \beta, n) = \binom{n}{t} \frac{B(t+\alpha, n-t+\beta)}{B(\alpha, \beta)}$$

$$= \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}}$$

for $t = 1, 2, \ldots, n$ and $\alpha, \beta > 0$.

Urn Models

Urn containing:

1. α red balls
2. β black balls

Draw n times:

1. **With Replacement:** Binomial

$$Pr[\#\text{ of red draws} = t] = \binom{n}{t}\left(\frac{\alpha}{\alpha+\beta}\right)^t\left(\frac{\beta}{\alpha+\beta}\right)^{n-t}$$

2. **With Duplication:** Beta-Binomial

$$Pr[\#\text{ of red draws} = t] = \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}}$$

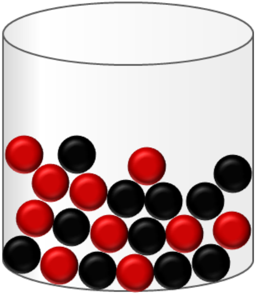

Figure 4: Urn with α red balls and β black balls.

Asymptotics

For large $\alpha + \beta$,

$$\mathrm{BB}(\alpha, \beta, n) \sim \mathrm{B}\left(n, \frac{\alpha}{\alpha+\beta}\right).$$

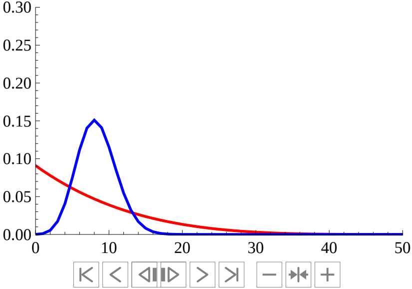

Figure 5: $\mathrm{BB}(\alpha, 10, 50)$ and $\mathrm{B}(50, \alpha/(\alpha+10))$ for $\alpha = 1, \ldots, 50$.

For large n,

$$\mathrm{BB}(\alpha, \beta, n) \sim \mathrm{Beta}(\alpha, \beta).$$

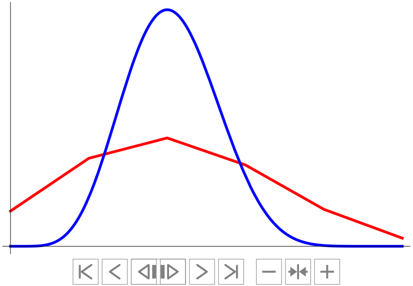

Figure 6: $\mathrm{BB}(7, 10, n)$ and $\mathrm{Beta}(7, 10)$ for $n = 5, \ldots, 50$.

Moments $M_m(X)$ of Probability Distributions

Let F be a probability distribution on $[0,1]$. If $X \sim F$, then the m^{th} **moment** of X is defined by

$$M_m(X) = \mathbb{E}[X^m] = \int_0^1 x^m dF(x).$$

In particular,

$$\text{Mean}(X) = \mathbb{E}[X] = M_1(X)$$

and

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = M_2(X) - M_1(X)^2\,.$$

Higher Order Moments

(Knoblauch [4]) If $X \sim \mathrm{B}(n,p)$, then

$$M_m(X) = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} p^i \, n^{\underline{i}} \, .$$

If $Y \sim \mathrm{Beta}(\alpha, \beta)$, then

$$M_m(Y) = \frac{\mathrm{B}(\alpha + m, \beta)}{\mathrm{B}(\alpha, \beta)} \, .$$

Theorem. If $Z \sim \mathrm{BB}(\alpha, \beta, n)$, then

$$M_m(Z) = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} \frac{\alpha^{\overline{i}}}{(\alpha+\beta)^{\overline{i}}} \, n^{\underline{i}} \, .$$

Completely Monotone Sequences

Given a finite or infinite numerical sequence $a_1, a_2, \ldots,$ the **forward difference operator** Δ is defined by

$$\Delta a_i = a_{i+1} - a_i \, .$$

A sequence is said to be **completely monotone** if

$$(-1)^r \Delta^r a_i = \sum_{j=0}^{r} (-1)^j \binom{r}{j} a_{i+j} \geq 0$$

for all $i, r \in \mathbb{N}$.

Example

The sequence $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$ is completely monotone:

$$\Delta^0 a_n = \left\{ 1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \tfrac{1}{5}, \quad \dots \right\}$$

$$\begin{aligned} \Delta^1 a_n &= \left\{ \tfrac{1}{2} - 1, \quad \tfrac{1}{3} - \tfrac{1}{2}, \quad \tfrac{1}{4} - \tfrac{1}{3}, \quad \tfrac{1}{5} - \tfrac{1}{4}, \quad \dots \right\} \\ &= \left\{ -\tfrac{1}{2}, \quad -\tfrac{1}{6}, \quad -\tfrac{1}{12}, \quad -\tfrac{1}{20}, \quad \dots \right\} \end{aligned}$$

$$\begin{aligned} \Delta^2 a_n &= \left\{ -\tfrac{1}{6} - \left(-\tfrac{1}{2}\right), \quad -\tfrac{1}{12} - \left(-\tfrac{1}{6}\right), \quad -\tfrac{1}{20} - \left(-\tfrac{1}{12}\right), \quad \dots, \quad \dots \right\} \\ &= \left\{ \tfrac{1}{3}, \quad \tfrac{1}{12}, \quad \tfrac{1}{30}, \quad \dots, \quad \dots \right\} \end{aligned}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\Delta^r a_n = \left\{ \frac{(-1)^r}{r+1}, \quad \frac{(-1)^r}{(r+1)(r+2)}, \quad \dots, \quad \frac{(-1)^r}{(r+1)^{\overline{n}}}, \quad \dots \right\}$$

Moment Sequences are Completely Monotone

- Given a distribution F on $[0,1]$ and $X \sim F$,

$$(-1)^r \Delta^r M_m(X) = \mathbb{E}[X^m (1-X)^r] \geq 0$$

for all $r \in \mathbb{N}$, i.e., moment sequences $\{M_r(X)\}$ are completely monotone.

- Conversely, every completely monotone sequence $\{a_r\}$ with $a_1 = 1$ coincides with the moment sequence of a unique probability distribution. (Hausdorff [3], Widder [6])

Bernstein's Theorem

Let f be a continuous function on the interval $[0, 1]$ then

$$f(t) = \lim_{n\to\infty} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} t^k (1-t)^{n-k}$$

uniformly on the interval $[0, 1]$.

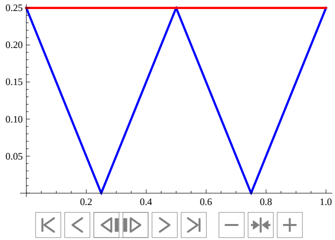

Figure 7: Bernstein polynomials converging to a sawtooth function on $[0, 1]$.

Corollaries

- Beta-Binomial version of Bernstein's Theorem:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{B}(k+\alpha, n-k+\beta)}{\mathrm{B}(\alpha,\beta)} h\left(\frac{k}{n}\right) = \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} h(t)\, dt$$

- Note that $f(x) = x^\lambda$ gives

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \left(\frac{k}{n}\right)^\lambda = \frac{M_\lambda(X)}{n^\lambda}$$

where $X \sim \mathrm{B}(n,t)$.

Main Binomial Sum

We consider

$$S_n(\alpha,\beta,\gamma,\lambda,\mu) = \sum_{k=0}^{n} \frac{\binom{\alpha+k-1}{k}\binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}} \left[\frac{\binom{\gamma+n-1}{n-k}}{\binom{\gamma+\lambda+n-1}{n-k}}\right]^{\mu}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha,\beta)} \left(\frac{B(n+\gamma,\lambda)}{B(k+\gamma,\lambda)}\right)^{\mu} .$$

Asymptotics Theorem

Theorem.

$$\lim_{n\to\infty} \sum_{k=0}^{n} \underbrace{\binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha,\beta)}}_{\mathrm{BB}(\alpha,\beta,n)} \underbrace{\left(\frac{B([k+\gamma]+[n-k],\lambda)}{B(k+\gamma,\lambda)}\right)^{\mu}}_{\substack{X \sim \mathrm{Beta}(k+\gamma,\lambda) \\ M_{(n-k)\mu}(X)}} = \underbrace{\frac{B(\alpha+\lambda\mu,\beta)}{B(\alpha,\beta)}}_{\substack{Y \sim \mathrm{Beta}(\alpha,\beta) \\ M_{\lambda\mu}(Y)}}$$

Remark. For large n,

$$\frac{\mathrm{B}(n+\gamma,\lambda)}{\mathrm{B}(k+\gamma,\lambda)} \approx \left(\frac{k}{n}\right)^{\lambda}$$

so the LHS of the theorem is asymptotically equal to the beta-binomial version of Bernstein's Theorem with $h(t) = t^{\lambda\mu}$.

Complete Monotonicity Theorem

- Given $X \sim \mathrm{B}(n,p)$ and $p \in (\frac{1}{2}, 1]$,

$$b_n(\lambda) = M_\lambda(X)/n^\lambda$$

  is not completely monotone for all $\lambda \in \mathbb{N}$.

- Given $Y \sim \mathrm{BB}(\alpha,\beta,n)$ and $\alpha > \beta$,

$$bb_n(\lambda) = M_\lambda(Y)/n^\lambda$$

  is not completely monotone for all $\lambda \in \mathbb{N}$.

- However, $S_n(\alpha,\beta,\gamma,\lambda,1) \sim bb_n(\lambda)$ is completely monotone for all $\lambda \in \mathbb{N}$.

Inequality Theorem

For every $n \in \mathbb{N}$,

$$S_n(\alpha, \beta + \lambda, \alpha, -\gamma, \mu)^{\nu} \cdot S_n\left(\beta, \alpha + \lambda, \beta, \gamma, \nu\right)^{\mu} \geq 1,$$

where

$$\alpha, \beta > 0,$$
$$\lambda \geq 0,$$
$$\mu > 1 \quad \left(\frac{1}{\mu} + \frac{1}{\nu} = 1\right), \text{ and}$$
$$\gamma \in (-\beta, \min\{0, \alpha - \beta\}) \cup (\max\{0, \alpha - \beta\}, \alpha).$$

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